## ON MINIMAL ZERO SEQUENCES WITH LARGE CROSS NUMBER

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### 1. Introduction and Main result

Throughout this section, let $G$ be a finite abelian group, which will be written additively.

Let $S = (g_1, \ldots, g_l)$ be a sequence of elements of $G$. Then $|S| = l$ denotes the *length* of $S$ and

$$k(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord}(g_i)}$$

its *cross number*. We say that $S$ is a *zero sequence*, if $\sum_{i=1}^{l} g_i = 0$ and that $S$ is *zero free*, if $\sum_{i \in I} g_i \neq 0$ for all $\emptyset \neq I \subseteq \{1, \ldots, l\}$. Furthermore, $S$ is called a *minimal zero sequence*, if it is a zero sequence and each proper subsequence is zero free.

Suppose $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ where $C_{n_1}, \ldots, C_{n_r}$ are cyclic groups of prime power order and let $\exp(G)$ denote the exponent of $G$. Investigations of the following invariants are motivated mainly by arithmetical problems in Krull domains (cf. [Ch]):

$$W(G) = \{k(S) \mid S \text{ is a minimal zero sequence in } G\},$$
$$K(G) = \exp(G) \max W(G),$$
$$k(G) = \max\{k(S) \mid S \text{ is a zero free sequence in } G\},$$
$$k^*(G) = \sum_{i=1}^{r} \frac{n_i - 1}{n_i}$$

and

$$K^*(G) = 1 + \exp(G) k^*(G) \ .$$

It is easy to see that

$$K^*(G) \leq 1 + \exp(G) k(G) \leq K(G) \qquad (1)$$

(cf. [G-S2; Lemma 1]). For $p$-groups and some further series of groups $K^*(G) = K(G)$ holds (see the references). Up to now there is known no group with $K^*(G) < K(G)$. Apart from the case of prime cyclic groups, we have almost no information about the structure of zero free sequences $S$ (resp. minimal zero sequences) with large cross numbers i.e., with $k(S)$ close to $k(G)$. In this paper we tackle the question about the structure in the very special case that $G$ is a direct sum of two elementary $p$-groups. Before we can state our main result we need a further definition.

Let $\rho(G)$ denote the smallest integer $l$ such that every sequence $S$ in $G$ with $\exp(G)k(S) \geq l$ contains a non-empty zero sequence $S' \subseteq S$ with $k(S') \leq 1$.

For every prime $p$ we have $\rho(C_p) = p$ and $\rho(C_p^2) = 3p - 2$. If $p$ is the minimal prime dividing $|G|$, then

$$1 + \exp(G)k(G) \leq \rho(G) \leq \frac{\exp(G)}{p}|G| . \tag{2}$$

Proofs may be found in [G-S1].

**Theorem.** *Let $G = C_p^r \oplus C_q^s$ with integers $r, s \in \mathbb{N}_+$ and primes $p, q$ with $p > max\{q, \rho(C_q^s) - (s-1)(q-1)\}$. Let $S$ be a zero free sequence in $G$ with*

$$k(S) \geq r\frac{p-1}{p} + s\frac{q-1}{q} - \frac{q-2}{pq} .$$

*Then $S = A \cup B$ where $A$ is a sequence in $C_p^r$ with $|A| = r(p-1)$ and $B$ is a sequence in $C_q^s$ with $|B| = s(q-1)$.*

This result describes the structure of $S$ and yields immediately the following corollary.

**Corollary 1.** *Let $G$ be as above. Then $K^*(G) = K(G)$.*

*Proof.* The Theorem implies that

$$k(G) \leq r\frac{p-1}{p} + s\frac{q-1}{q} .$$

Hence by (1) it follows that $k^*(G) = k(G)$. Therefore, a simple calculation (or Corollary 1 in [G-S2]) gives the assertion. $\square$

Recall that *Davenport's constant* $D(G)$ of $G$ is defined as

$$D(G) = \max\{|S| \mid S \text{ is a minimal zero sequence in } G\} .$$

If $G$ is as in the Theorem, then the exact value of $D(G)$ is unknown and there is no information at all about the structure of long minimal zero sequences. Davenport's constant of $G = C_p \oplus C_2^s$ was studied in [Ma].

By definition we have

$$W(G) \subseteq \left\{ \frac{i}{\exp(G)} \,|\, 2 \le i \le K(G) \right\} . \tag{3}$$

If $G$ is a $p$-group for some odd prime $p$, then equality holds. This was proved in [C-G]. The next corollary gives the first example of groups of odd order, for which the inclusion in (3) is strict.

**Corollary 2.** *Let $G$ be as above. Then*

$$W(G) \subseteq \left\{ \frac{i}{\exp(G)} \,|\, 2 \le i \le K(G) - q + 1 \text{ or } i = K(G) \right\} .$$

*Proof.* Let $S_0$ be a minimal zero sequence distinct to $(0)$. Suppose that $k(S_0) \ge \frac{1}{pq}(K(G) - q + 2)$. Obviously, $S_0 = g \cup S$ for some $g \in G$ with $\operatorname{ord}(g) = \exp(G) = pq$ and some zero free sequence $S$. Therefore

$$\begin{aligned} k(S) &= k(S_0) - \frac{1}{pq} \\ &\ge \frac{1}{pq}(K(G) - q + 2) - \frac{1}{pq} \\ &= \frac{1}{pq}(K^*(G) - (q-2)) - \frac{1}{pq} \\ &= \frac{1}{pq}(rq(p-1) + sp(q-1) - (q-2)) \end{aligned}$$

Then the Theorem implies that $k(S) = r\frac{p-1}{p} + s\frac{q-1}{q}$ whence $k(S_0) = \frac{1}{pq}K(G)$. □

## 2. Proof of the Theorem

We start with a simple lemma.

**Lemma.** *Let $G$ be a finite abelian group, $H \le G$ a subgroup and $\pi \colon G \to G/H$ the canonical epimorphism. Let $S$ be a sequence in $G$ and $\overline{S} = \pi(S)$ its image.*

**a)** *Suppose $k(\overline{S}) \ge k(G/H) + \frac{1}{\exp(G/H)}$. Then there exists a non-empty subsequence $S_0 \subseteq S$ with $\sum_{g \in S_0} g \in H$.*

**b)** *Suppose* $k(\overline{S}) \geq \frac{\rho(G/H)}{\exp(G/H)} + l$ *for some* $l \in \mathbb{N}_0$. *Then there exist disjoint subsequences* $S_0, \ldots, S_l$ *with* $\bigcup_{i=0}^{l} S_i \subseteq S$ *such that* $\sum_{g \in S_i} g \in H$ *for all* $0 \leq i \leq l$.

**c)** *Suppose* $S = S' \cup \bigcup_{i=0}^{l} S_i$ *such that* $h_i = \sum_{g \in S_i} g \in H$ *for all* $0 \leq i \leq l$. *Then* $S^* = S' \cup (h_0, \ldots, h_l)$ *is a sequence in* $G$ *with* $k(S^*) \geq k(S') + \frac{l+1}{\exp(H)}$. *Moreover, if* $S$ *is a zero sequence, a minimal zero sequence or zero free, then the same is true for* $S^*$.

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*Proof.* **a)** By [G-S1; Lemma 1] there exists a non-empty subsequence $T \subseteq \overline{S}$ with sum zero. Hence $T$ has a preimage $S_0 \subseteq S$ with $\sum_{g \in S_0} g \in H$.

**b)** We proceed by induction on $l$. Suppose $l = 0$. Taking (2) into account the assertion follows from **a)**. To do the induction step one just has to use the very definition of $\rho(G/H)$.

**c)** Straightforward. □

*Proof of the Theorem.* Set $S = A \cup B \cup C$ where $A$ is a sequence in $C_p^r$ with $|A| = \alpha$, $B$ is a sequence in $C_q^s$ with $|B| = \beta$ and $C$ is a sequence in $G \setminus (C_p^r \cup C_q^s)$ with $|C| = \gamma$.

Since $D(C_p^r) = 1 + r(p-1)$ (cf. [Al; section 6.1]) and since $S$ is zero free, we infer that $\alpha \leq r(p-1)$. An analogous argument shows that $\beta \leq s(q-1)$. Suppose $\gamma = 0$. If $\alpha \leq r(p-1) - 1$, then

$$\frac{r(p-1)-1}{p} + s\frac{q-1}{q} \geq k(S) \geq r\frac{p-1}{p} + s\frac{q-1}{q} - \frac{q-2}{pq},$$

a contradiction. If $\beta \leq s(q-1) - 1$, then

$$r\frac{p-1}{p} + \frac{s(q-1)-1}{q} \geq k(S) \geq r\frac{p-1}{p} + s\frac{q-1}{q} - \frac{q-2}{pq}$$

which implies that $q - 2 \geq p$, contradicting our assumption on $p$ and $q$. Hence, if $\gamma = 0$, then the assertion follows.

Assume to the contrary, that $\gamma \geq 1$. We distinguish two cases.

**Case 1.** $k(B) = s\frac{q-1}{q}$. Let $H = C_q^s, \pi \colon G \to G/H = C_p^r$ the canonical epimorphism, $\overline{A} = \pi(A)$ and $\overline{C} = \pi(C)$. Then $k(\overline{A} \cup \overline{C}) = \frac{\alpha+\gamma}{p}$. Clearly,

$$\frac{\alpha}{p} + \frac{\gamma}{pq} = k(S) - s\frac{q-1}{q} \geq r\frac{p-1}{p} - \frac{q-2}{pq}$$

and thus

$$\alpha q + \gamma \geq rq(p-1) - (q-2).$$

Since

$$q(\alpha+\gamma) = \alpha q + \gamma + (q-1)\gamma \geq \alpha q + \gamma + q - 1 \geq rq(p-1) + 1\ ,$$

we infer that

$$pk(\overline{A}\cup\overline{C}) = \alpha + \gamma \geq r(p-1) + \frac{1}{q} = pk(C_p^r) + \frac{1}{q}\ .$$

However, because $pk(\overline{A}\cup\overline{C}) \in \mathbb{N}_+$ and $pk(C_p^r) \in \mathbb{N}_+$ it follows that

$$pk(\overline{A}\cup\overline{C}) \geq pk(C_p^r) + 1\ .$$

By Lemma **a)** there exists a subsequence $S_0 \subseteq A \cup C$ with $b = \sum_{g\in S_0} g \in C_q^s$. Then $T = b \cup B$ is a zero free sequence in $C_q^s$ with

$$k(T) = \frac{1}{q} + k(B) > k(C_q^s)\ ,$$

a contradiction.

**Case 2.** $k(B) < s\frac{q-1}{q}$. Let $H = C_p^r, \pi\colon G \to G/H = C_q^s$ the canonical epimorphism, $\overline{B} = \pi(B)$ and $\overline{C} = \pi(C)$. Obviously, we have $k(\overline{B}\cup\overline{C}) = \frac{\beta+\gamma}{q}$. Define

$$l = \left[pk(S) - \alpha - \beta\frac{p-1}{q} - \frac{\rho(C_q^s)}{q}\right]\ .$$

We verify in a moment that $l$ is a non-negative integer. Since

$$k(S) = \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{pq}$$

it follows that

$$\frac{\gamma}{q} = pk(S) - \beta\frac{p}{q} - \alpha$$

and hence

$$\begin{aligned} k(\overline{B}\cup\overline{C}) = \frac{\beta+\gamma}{q} &= pk(S) - \alpha - \beta\frac{p-1}{q} \\ &\geq \frac{\rho(C_q^s)}{q} + l \\ &= \frac{\rho(G/H)}{\exp(G/H)} + l\ . \end{aligned}$$

By Lemma **b)** there exist sequences $S_0, \dots, S_l$ with $\bigcup_{i=0}^{l} S_i \subseteq B \cup C$ such that $h_i = \sum_{g \in S_i} g \in H$ for $0 \le i \le l$. By Lemma **c)** the sequence $S^* = A \cup (h_0, \dots, h_l)$ is zero free and $k(S^*) \ge \frac{\alpha}{p} + \frac{l+1}{p}$. However,

$$\begin{aligned}
\frac{\alpha + l + 1}{p} &\ge k(S) - \beta\frac{p-1}{pq} - \frac{\rho(C_q^s)}{pq} \\
&\ge r\frac{p-1}{p} + s\frac{q-1}{q} - \frac{q-2}{pq} - \beta\frac{p-1}{pq} - \frac{\rho(C_q^s)}{pq} \\
&\ge r\frac{p-1}{p} + \frac{ps(q-1) - (q-2) - (s(q-1)-1)(p-1) - \rho(C_q^s)}{pq} \\
&> r\frac{p-1}{p} = k(C_p^r) ,
\end{aligned}$$

where the last inequality follows from the hypothesis $p > \rho(C_q^s) - (s-1)(q-1)$. Since $\alpha \le r(p-1)$, this calculation shows in particular that $l$ is non-negative. Furthermore, it contradicts the zero freeness of $S^*$. □

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