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On Quasi Divisor Theories and Systems of Valuations

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Abstract. We study the relationship between divisor theories and systems of valuations, and characterize monoids with quasi divisor theories of finite character by systems of essential valuations. Throughout, we avoid ideal theory but use divisor theoreticai methods.

1. The notion of divisibility, which is fundamental for all arithmetical investigations, is a purely multiplicative one, and there are several fundamental papers making allowance for this point of view, cf. [8], [12], [1].

There are two different concepts for a description of the arithmetic of a domain without unique factorization: ideals and divisors. The ideal-theoretic point of view was first described by A. CLIFFORD [2] and later on in [7] and [1], where, more generally, ideal systems (in particular t-ideals) were considered. The first axiomatic treatment of divisors was given by L. SKULA in [12]; however, in order to prove uniqueness of a divisor theory, he was forced to fall back upon ideal-theoretic concepts and to refer to [2]. In [6], the second author stressed the connection between divisor theories and valuations and succeeded in giving a proof for the uniqueness of a divisor theory avoiding ideal-theoretic tools.

It is the aim of this paper to do the same for quasi divisor theories as introduced in [1] and investigated in [10] and [4]. We start with a proof of Lorenzen's Realization Theorem for GCD-monoids (cf. [9; Theorem 1.9]) which works entirely in the language of monoids and monoid homomorphisms. For convenience of the reader we explicitely formulate the connection between the language of quasi divisor theories and the language of t-ideals in Proposition 3 (cf. [7; II §2 Prop. 7]).

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The central topics are the following: the uniqueness theorem for quasi divisor theories (Theorem 2) cf. [7; II §3 Corollary of Theorem 3] and [1; Theorem 4]); a description of quasi divisor theories of finite character by valuations (Theorem 3) first proved in [4]; a new criterion for the minimality of families of defining essential valuations of finite character (Corollary 1); and a fresh approach towards the Realization Theorem of Krull–Kaplansky–Jaffard–Ohm (Corollary 2).

2. All monoids in this paper are assumed to be commutative and cancellative, and they are written multiplicatively. Our main reference for the theory of monoid homomorphisms is [3]; for the notions of divisibility theory in monoids we refer to [5; §6]. For a monoid H, we denote by $\mathcal{Q}(H)$ a quotient group of H with $H \subset \mathcal{Q}(H)$ and by $H^\times$ the group of invertible elements of H; H is called *reduced* if $H^\times = \{1\}$.

Let $\varphi: H \to D$ be a monoid homomorphism. We set

$$H_\varphi = \{u^{-1}v \in \mathcal{Q}(H) \mid u, v \in H, \varphi(u) \mid \varphi(v)\};$$

furthermore, φ is said to be

a) a *divisor homomorphism*, if $u, v \in H$ and $\varphi(u) \mid \varphi(v)$ implies $u \mid v$.

b) *essential*, if $u, v \in H$ and $\varphi(u) \mid \varphi(v)$ implies $u \mid vs$ for some $s \in \varphi^{-1}(D^\times)$.

Definition 1. Let H be a monoid.

a) H is called a GCD-*monoid* if any two elements of H possess a unique greatest common divisor in H (equivalently: H is reduced, and any two elements of H possess a greatest common divisor in H). If H is a GCD-monoid, then any finitely many elements $a_1, \dots, a_n \in H$ possess a unique greatest common divisor in H, denoted by $(a_1, \dots, a_n)_H$.

b) A homomorphism of GCD-monoids φ: $H \to D$ is called a GCD-*homomorphism* if $\varphi((a, b)_H) = (\varphi(a), \varphi(b))_D$ for all $a, b \in H$.

c) H is called a *valuation monoid* if for any $a, b \in H$ we have $a \mid b$ or $b \mid a$ (equivalently: if $x \in \mathcal{Q}(H) \setminus H$, then $x^{-1} \in H$). Clearly, any reduced valuation monoid is a GCD-monoid.

d) A *valuation* of H is a surjective homomorphism φ: $H \to V$ into a reduced valuation monoid V.

Let $(D_i)_{i \in I}$ be a family of GCD-monoids, and

$$D' = \coprod_{i \in I} D_i \subset \prod_{i \in I} D_i = D.$$

Then D and D' are GCD-monoids, and for any $(a_i)_{i\in I}$, $(b_i)_{i\in I}\in D$ we have

$$((a_i)_{i\in I},(b_i)_{i\in I})_D=((a_i,b_i)_{D_i})_{i\in I}.$$

In particular, the injection $D'\hookrightarrow D$ and the projections $D\to D_i$ are GCD-homomorphisms.

Proposition 1. *Let D be a* GCD-*monoid, V a reduced valuation monoid and $\varphi\colon D\to V$ a monoid homomorphism. Then φ is essential if and only if φ is a* GCD-*homomorphism.*

Proof. Let φ be essential, $a,b\in D$ and $d=(a,b)_D$. We may assume that $\varphi(a)|\varphi(b)$; then there exists some $z\in D$ such that $\varphi(z)=1$ and $a|bz$, say $bz=ac$ for some $c\in D$. This implies

$$dz=(az,bz)_D=(az,ac)_D=a(z,c)_D,$$

and consequently $a|dz$, whence $\varphi(a)|\varphi(dz)=\varphi(d)$. On the other hand, $d|a$ implies $\varphi(d)|\varphi(a)$ and therefore $\varphi(d)=\varphi(a)=(\varphi(a),\varphi(b))_V$.

Assume now that φ is a GCD-homomorphism, and let $a,b\in D$ be such that $\varphi(a)|\varphi(b)$. If $d=(a,b)_D$, then $\varphi(d)=(\varphi(a),\varphi(b))_V=\varphi(a)$. Let $z\in D$ be such that $a=dz$; then $a|bz$, and $\varphi(d)\varphi(z)=\varphi(a)=\varphi(d)$ implies $\varphi(z)=1$. ■

Definition 2. Let $\boldsymbol{\rho}=(\rho_i\colon H\to D_i)_{i\in I}$ be a family of monoid homomorphisms. Then we define

$$\rho=\prod_{i\in I}\rho_i\colon H\to\prod_{i\in I}D_i$$

by

$$\rho(a)=(\rho_i(a))_{i\in I}.$$

The family $\boldsymbol{\rho}$ resp. the homomorphism ρ is said to be *of finite character* if

$$\rho(H)\subset\coprod_{i\in I}D_i$$

(equivalently: For any $z\in H$, we have $\rho_i(z)=1$ for all but finitely many $i\in I$). The family $\boldsymbol{\rho}$ is said to be a *defining family* for H if

$$H=\bigcap_{i\in I}H_{\rho_i}$$

(equivalently, ρ is a divisor homomorphism; cf. [3; Proposition 3.2]).

Definition 3. Let D be a GCD-monoid. By a *realization* of D we mean a defining family of essential valuations of D. D is called *of finite character* if it possesses a realization of finite character.

Theorem 1 (Lorenzen's Realization Theorem). *Every* GCD-*monoid has a realization.*

Proof. Let D be a GCD-monoid. For $S \subset D$, we set

$$S^{-1}\cdot D = \{s^{-1}a \mid s \in S, a \in D\} \subset \mathcal{Q}(D).$$

A submonoid $S \subset D$ is called divisor closed if $a \in S$, $b \in D$ and $b \mid a$ implies $b \in S$. Since D is a GCD-monoid, every $x \in \mathcal{Q}(D)$ has a unique representation $x = a^{-1}b$, where $a, b \in D$ and $(a, b)_D = 1$; if $S \subset D$ is a divisor closed submonoid, then $x \in S^{-1}\cdot D$ if and only if $a \in S$, which implies $(S^{-1}\cdot D)^{\times} \cap D = S$ and that $D \hookrightarrow S^{-1}\cdot D$ is an essential homomorphism.

If $1 \neq a \in D$, then a standard argument using Zorn's Lemma shows that there exists a maximal divisor closed submonoid $S_a \subset D$ such that $a \notin S_a$. By construction,

$$D = \bigcap_{1 \neq a \in D} S_a^{-1}\cdot D,$$

and it is sufficient to prove that the monoids $S_a^{-1}\cdot D$ are valuation monoids; for then the canonical family

$$(D \hookrightarrow S_a^{-1}\cdot D \to (S_a^{-1}\cdot D)/(S_a^{-1}\cdot D)^{\times})_{1 \neq a \in D}$$

is a realization of D.

Let $1 \neq a \in D$ be given, and suppose that $S_a^{-1}\cdot D$ is not a valuation monoid. Then there exist elements $b, c \in D \setminus S_a$ such that $(b, c)_D = 1$. By the maximality of S_a, we obtain $a \mid b^n u$ and $a \mid c^m v$ for some $m, n \in \mathbb{N}$ and $u, v \in S_a$. Putting $r = \max(m, n)$ and $w = uv \in S_a$, we infer

$$a \mid (b^r w, c^r w)_D = w$$

and hence $a \in S_a$, a contradiction. ■

3. Now we define the notion of a quasi divisor theory and prove some results concerning the extension of valuations which imply the uniqueness of quasi divisor theories; cf. [1; Theorem 1 and Cor. 1] and [4; Prop. 3.4].

Definition 4. By a *quasi divisor theory* (QDT for short) we mean a divisor homomorphism $\partial\colon H \to D$ into a GCD-monoid D with the following property:

D) For any $a \in D$, there exist elements $u_1, \dots, u_n \in H$ such that $a = (\partial u_1, \dots, \partial u_n)_D$. A QDT $\partial: H \to D$ is called *of finite character* if D is of finite character.

Proposition 2. *Let $\partial: H \to D$ be a* QDT, *and let $\boldsymbol{\rho} = (\rho_i: D \to V_i)_{i \in I}$ be a realization of D; for $i \in I$, set $\partial_i = \rho_i \circ \partial: H \to V_i$. Then $\boldsymbol{\partial} = (\partial_i)_{i \in I}$ is a defining family of essential valuations of H. If $\boldsymbol{\rho}$ is of finite character, then so is $\boldsymbol{\partial}$.*

Proof. Clearly, if $\boldsymbol{\rho}$ is of finite character, then so ist $\boldsymbol{\partial}$. Since $(\prod_{i \in I} \rho_i)$ and ∂ are divisor homomorphisms, $(\prod_{i \in I} \partial_i) = (\prod_{i \in I} \rho_i) \circ \partial$ is also a divisor homomorphism, and therefore $\boldsymbol{\partial}$ is a defining family.

It remains to prove that the homomorphisms ∂_i are essential and surjective. We fix some $i \in I$, and we first prove the following:

(∗) For any $b \in D$, there exists some $w \in H$ such that $b \mid \partial w$ and $\rho_i(b) = \partial_i(w)$.

Indeed, if $b \in D$, then there exist $u_1, \dots, u_n \in H$ such that $b = (\partial u_1, \dots, \partial u_n)_D$. By Proposition 1, ρ_i is a GCD-homomorphism, and therefore

$$\rho_i(b) = (\rho_i(\partial u_1), \dots, \rho_i(\partial u_n))_{V_i} = (\partial_i(u_1), \dots, \partial_i(u_n))_{V_i} = \partial_i(u_\nu)$$

for some $1 \leqslant \nu \leqslant n$, and clearly $b \mid \partial u_\nu$, whence (∗) holds.

Since ρ_i is surjective, (∗) implies that ∂_i is surjective.

In order to prove that ∂_i is essential, let $u, v \in H$ be such that $\partial_i(u) \mid \partial_i(v)$. Since ∂_i is surjective, there exists some $z \in H$ such that $\partial_i(v) = \partial_i(uz)$. If $c = (\partial(v), \partial(uz))_D$, then

$$\rho_i(c) = (\rho_i \circ \partial(v), \rho_i \circ \partial(uz))_{V_i} = \partial_i(v) = \partial_i(uz),$$

which implies $b = c^{-1}\partial(uz) \in D$ and $\rho_i(b) = 1$. From $c \mid \partial(v)$ we obtain

$$\partial(u) \mid c^{-1}\partial(uz)\partial(v) = b\partial(v).$$

Now (∗) implies the existence of some $w \in H$ such that $b \mid \partial(w)$ and $\rho_i(b) = \partial_i(w) = 1$. Some $\partial(u) \mid \partial(v)\partial(w) = \partial(vw)$, we infer $u \mid uw$. ■

Proposition 3. *Let $\partial: H \to D$ be a* QDT, *$v_1, \dots, v_n \in H$, $d = (\partial v_1, \dots, \partial v_n)_D$ and $\{v_1, \dots, v_n\}_t$ the t-ideal generated by $v_1, \dots, v_n$. Then $\{v_1, \dots, v_n\}_t = \{x \in H \mid d \mid \partial x\}$.*

Proof. By definition we have

$$\{v_1, \dots, v_n\}_t = \bigcap_{\substack{y \in \mathscr{Q}(H) \\ \{v_1, \dots, v_n\} \subset yH}} yH = \bigcap_{\substack{u, v \in H \\ \{uv_1, \dots, uv_n\} \subset vH}} u^{-1}vH.$$

For all $u, v \in H$ it holds that $\{uv_1, \ldots, uv_n\} \subset vH$ if and only if $\partial v | d\partial u$. Hence we have to verify that for all $x \in H$ the following two conditions are equivalent:

i) $d | \partial x$.

ii) For all $u, v \in H$ $\partial v | d\partial u$ implies that $v | xu$.

Clearly, i) implies ii). Conversely, let $x \in H$ be given with $d \nmid \partial x$. Then, by Proposition 2, there exists an essential valuation $\varphi: D \to V$ such that $\varphi(d) \nmid \varphi(\partial x)$. Since φ is a GCD-homomorphism we infer that

$$\varphi(d) = (\varphi(\partial v_1), \ldots, \varphi(\partial v_n))_V = \varphi(\partial v_j)$$

for some $j \in \{1, \ldots, n\}$. There are $u_1, \ldots, u_m \in H$ with $d^{-1}\partial v_j = (\partial u_1, \ldots, \partial u_m)_D$ and hence

$$1 = \varphi(d^{-1}\partial v_j) = (\varphi(\partial u_1), \ldots, \varphi(\partial u_m))_V = \varphi(\partial u_i)$$

for some $i \in \{1, \ldots, m\}$. Thus $\partial v_j | d\partial u_i$ but $v_j \nmid xu_i$ since $\varphi(\partial v_j) = \varphi(d) \nmid \varphi(\partial x) = \varphi(\partial(xu_i))$. ■

Proposition 4. *Let $\partial: H \to D$ be a* QDT *and $\varphi: H \to V$ an essential valuation. Then there exists a unique essential valuation $\bar{\varphi}: D \to V$ such that $\bar{\varphi} \circ \partial = \varphi$.*

Proof. Passing from H to $H/H^\times$ and identifying the latter monoid with ∂H, we may assume that $H \subset D$ is a saturated submonoid (i.e. $H = D \cap \mathcal{Q}(H)$) and $\partial = (H \hookrightarrow D)$ (cf. [3; Lemma 2.4]). If $\bar{\varphi}: D \to V$ is an essential valuation satisfying $\bar{\varphi} | H = \varphi$, then $\bar{\varphi}$ is a GCD-homomorphism by Proposition 1. If $a \in D$ and $u_1, \ldots, u_n \in H$ are such that $a = (u_1, \ldots, u_n)_D$, then $\bar{\varphi}(a) = (\bar{\varphi}(u_1), \ldots, \bar{\varphi}(u_n))_V = (\varphi(u_1), \ldots, \varphi(u_n))_V$. Therefore $\bar{\varphi}$ is uniquely determined by φ.

For $a \in D$, choose $u_1, \ldots, u_n \in H$ such that $a = (u_1, \ldots, u_n)_D$, and define $\bar{\varphi}(a) = (\varphi(u_1), \ldots, \varphi(u_n))_V$. We must prove that this definition does not depend on the choice of $u_1, \ldots, u_n$. We may assume that $\varphi(u_1) | \varphi(u_\nu)$ for all $1 \leqslant \nu \leqslant n$, and then we must prove that $\varphi(u_1) | \varphi(u)$ for all $u \in H$ such that $a | u$ (in D). Let $u \in H$ such that $a | u$ (in D) be given, say $u = ab$, where $b \in D$. Let $v_1, \ldots, v_m \in H$ be such that $b = (v_1, \ldots, v_m)_D$ and $\varphi(v_1) | \varphi(v_j)$ for all $1 \leqslant j \leqslant m$; then

$$u = ab = (\{u_i v_j | 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\})_D.$$

For all i, j we have $\varphi(u_1 v_1) | \varphi(u_i v_j)$, and since φ is essential, there exists some $z \in H$ such that $\varphi(z) = 1$ and $u_1 v_1 | u_i v_j z$ for all i and j. Let

$y_{ij} \in H$ be such that $u_i v_j z = u_1 v_1 y_{ij}$; then

$$uz = (\{u_i v_j z \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\})_D = \\ = (\{u_1 v_1 y_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\})_D = u_1 v_1 c$$

for some $c \in D$. This implies $u_1 \mid uz$ (in D and hence also in H), and consequently $\varphi(u_1) \mid \varphi(uz) = \varphi(u)$.

If $a, b \in D$ and $u_1, \ldots, u_n$, $v_1, \ldots, v_m$ are such that $a = (u_1, \ldots, u_n)_D$ and $b = (v_1, \ldots, v_m)_D$, then $ab = (\{u_i v_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\})_D$ and therefore

$$\bar{\varphi}(ab) = (\{\varphi(u_i v_j) \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\})_V = \\ = (\varphi(u_1), \ldots, \varphi(u_n))_V (\varphi(v_1), \ldots, \varphi(v_m))_V = \bar{\varphi}(a)\bar{\varphi}(b).$$

Thus $\bar{\varphi}$ is a monoid homomorphism; clearly, $\bar{\varphi} \mid H = \varphi$ and $\bar{\varphi}$ is surjective.

It remains to prove that $\bar{\varphi}$ is essential. Let $a, b \in D$ be such that $\bar{\varphi}(a) \mid \bar{\varphi}(b)$, and choose elements $u_1, \ldots, u_n$, $v_1, \ldots, v_m \in H$ satisfying $a = (u_1, \ldots, u_n)_D$, $b = (v_1, \ldots, v_m)_D$, $\varphi(u_1) \mid \varphi(u_i)$ for $1 \leqslant i \leqslant n$ and $\varphi(v_1) \mid \varphi(v_j)$ for all $1 \leqslant j \leqslant m$. Then we have, for $1 \leqslant j \leqslant m$,

$$\varphi(u_1) = \bar{\varphi}(a) \mid \bar{\varphi}(b) = \varphi(v_1) \mid \varphi(v_j);$$

since φ is essential, there exists some $t \in H$ such that $u_1 \mid t v_j$, and consequently

$$(u_1, \ldots, u_n)_D \mid (t v_1, \ldots, t v_m, u_2, \ldots, u_n)_D \mid (t, u_2, \ldots, u_n)_D (v_1, \ldots, v_m)_D.$$

Putting $c = (t, u_2, \ldots, u_n)_D$, we obtain $a \mid bc$, and since $c \mid t$ and $\bar{\varphi}(t) = \varphi(t) = 1$, also $\bar{\varphi}(c) = 1$. ■

Proposition 5. *Let* $\partial: H \to D$ *be a* QDT *and* $(\varphi_i: H \to V_i)_{i \in I}$ *a defining family of essential valuations of* H. *For* $i \in I$, *let* $\bar{\varphi}_i: D \to V_i$ *be the (unique) essential valuation satisfying* $\bar{\varphi}_i \circ \partial = \varphi_i$. *Then* $(\bar{\varphi}_i)_{i \in I}$ *is a defining family of* D.

Proof. Let $a, b \in D$ be such that $\bar{\varphi}_i(a) \mid \bar{\varphi}_i(b)$ for all $i \in I$. Let $c \in D$ and $u, u_1, \ldots, u_n \in H$ be elements satisfying $ac = \partial u$ and $bc = (\partial u_1, \ldots, \partial u_n)_D$. For any $i \in I$, $\bar{\varphi}_i$ is a GCD-homomorphism by Proposition 1, and therefore

$$\varphi_i(u) = \bar{\varphi}_i(\partial u) = \bar{\varphi}_i(ac) \mid \bar{\varphi}_i(bc) = (\bar{\varphi}_i(\partial u_1), \ldots, \bar{\varphi}_i(\partial u_n))_{V_i} = \\ = (\varphi_i(u_1), \ldots, \varphi_i(u_n))_{V_i},$$

whence $\varphi_i(u) \mid \varphi_i(u_\nu)$ for all $1 \leqslant \nu \leqslant n$ and all $i \in I$. Since $(\varphi_i)_{i \in I}$ is a defining family of H, we infer $u|u_\nu$ for $1 \leqslant \nu \leqslant n$, and consequently $ac|bc$, whence $a|b$. ■

Theorem 2 (Uniqueness of QDTs). *Let* $\partial: H \to D$ *and* $\partial': H \to D'$ *be* QDTs. *Then there exists a unique isomorphism* $\Phi: D \to D'$ *satisfying* $\partial' = \Phi \circ \partial$.

Proof. By general reasons, it is enough to prove that there exists a unique GCD-homomorphism $\Phi: D \to D'$ satisfying $\partial' = \Phi \circ \partial$. If Φ is such a GCD-homomorphism, $a \in D$ and $u_1, \ldots, u_n \in H$ are such that $a = (\partial u_1, \ldots, \partial u_n)_D$, then $\Phi(a) = (\Phi(\partial u_1), \ldots, \Phi(\partial u_n))_{D'} = (\partial' u_1, \ldots, \partial' u_n)_{D'}$; consequently, Φ is uniquely determined.

In order to prove the existence of Φ, let $(\rho_i: D' \to V_i)_{i \in I}$ be a realization of D. By Proposition 1, all ρ_i are GCD-homomorphisms, and hence

$$\rho = \prod_{i \in I} \rho_i: D' \to V = \prod_{i \in I} V_i$$

is a GCD-homomorphism. For $i \in I$, $\varphi_i \doteq \rho_i \circ \partial'$ is an essential valuation by Proposition 2, and by Proposition 3 there exists a unique essential valuation Φ_i: $D \to V_i$ satisfying $\Phi_i \circ \partial = \varphi_i$. By Proposition 1, each Φ_i is a GCD-homomorphism, and hence $\Phi = \prod_{i \in I} \Phi_i$ is a GCD-homomorphism. Now we assert that $\Phi(D) \subset \rho(D')$; then $\rho^{-1} \circ \Phi: D \to D'$ is the desired GCD-homomorphism.

Indeed, let $a \in D$ and $u_1, \ldots, u_n \in H$ be such that $a = (\partial u_1, \ldots, \partial u_n)_D$; then

$$\Phi(a) = (\Phi(\partial u_1), \ldots, \Phi(\partial u_n))_V = (\rho(\partial' u_1), \ldots, \rho(\partial' u_n))_V = \rho((\partial' u_1, \ldots, \partial' u_n)_{D'}) \in \rho(D). \quad ■$$

4. Next we characterize monoids having a QDT of finite character by families of essential valuations, cf. [4; Theorem 3.8]. For the relevance of this result, cf. [4; §5].

Theorem 3. *Let H be a monoid. Then the following assertions are equivalent*:

a) *H possesses a* QDT *of finite character.*

b) *There exists a defining family of essential valuations of finite character of H.*

Proof. a)⇒b) follows from Proposition 2 and Theorem 1.

b)⇒a) Let $(\varphi_i)_{i \in I}$ be a defining family of essential valuations of finite character, and set

$$\varphi = \prod_{i\in I} \varphi_i : H \to V = \coprod_{i\in I} V_i \subset \prod_{i\in I} V_i.$$

Let $\mathscr{E}(H)$ be the set of all finite subsets $\emptyset \neq A \subset H$. We make $\mathscr{E}(H)$ into a semigroup by setting $AB = \{ab \mid a\in A, b\in B\}$. For $A = \{a_1, \dots, a_n\} \in \mathscr{E}(H)$, we define

$$\rho_0(A) = (\varphi(a_1), \dots, \varphi(a_n))_V \in V.$$

Then ρ_0 is a semigroup homomorphism and induces a monomorphism

$$\rho : D = \mathscr{E}(H)/\equiv \; \to V,$$

where the congruence relation $\equiv$ on $\mathscr{E}(H)$ is defined by

$$A \equiv B \quad \text{if and only if} \quad \rho_0(A) = \rho_0(B).$$

For $A \in \mathscr{E}(H)$, we denote by $[A] \in D$ the equivalence class of A, and for $A = \{a_1, \dots, a_n\}$, we set $[a_1, \dots, a_n] = [A] \in D$.

We define $\partial : H \to D$ by $\partial(a) = [a]$; then ∂ is a monoid homomorphism, $\rho \circ \partial = \varphi$, and we shall prove that ∂ is a QDT of finite character.

CLAIM 1: ∂ is a divisor homomorphism.

If $u, v \in H$ and $\partial u \mid \partial v$, then $\rho\circ\partial(u) \mid \rho\circ\partial(v)$, whence $\varphi(u) \mid \varphi(v)$ and hence $u \mid v$, since φ is a divisor homomorphism.

CLAIM 2: ρ is a divisor homomorphism.

To prove this, we must show that $A, B \in \mathscr{E}(H)$ and $\rho_0(A) \mid \rho_0(B)$ (in V) implies $[A] \mid [B]$ (in D). Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be given; for $i \in I$, let $\nu(i) \in \{1, \dots, n\}$ be such that

$$\varphi_i(a_{\nu(i)}) = (\varphi_i(a_1), \dots, \varphi_i(a_n))_{V_i}.$$

Now we assert that there exists a family $(z_i)_{i\in I}$ in H having the following properties:

$$a_{\nu(i)} \mid b_\mu z_i \quad \text{and} \quad a_{\nu(i)} \mid a_k z_i \quad \text{for all} \quad i \in I, 1 \leqslant \mu \leqslant m, 1 \leqslant k \leqslant n;$$
$$\varphi(z_i) = 1 \quad \text{for all} \quad i \in I;$$
$$\{z_i \mid i \in I\} \quad \text{is a finite set.}$$

For $1 \leqslant l \leqslant n$, set $I_l = \{i \in I \mid \nu(i) = l\}$. If $i \in I_l$, $1 \leqslant \mu \leqslant m$, $1 \leqslant k \leqslant n$, then $\varphi_i(a_{\nu(i)}) \mid \varphi_i(b_\mu)$ and $\varphi_i(a_{\nu(i)}) \mid \varphi_i(a_k)$; since φ_i is essential, there exists an element $z_i \in H$ such that $\varphi_i(z_i) = 1$, $a_{\nu(i)} \mid b_\mu z_i$ and $a_{\nu(i)} \mid a_k z_i$. Since $(\varphi_i)_{i\in I}$

is a family of finite character, one fixed z_{i_0} does the job for all but finitely many $i \in I$, and therefore we may assume that the set $\{z_i | i \in I\}$ is finite.

Having constructed our family $(z_i)_{i \in I}$, let $y_{i,\mu} \in H$ be such that $b_\mu z_i = a_{\nu(i)} y_{i,\mu}$; then the set $\{y_{i,\mu} | i \in I, 1 \leqslant \mu \leqslant m\}$ is also finite, and we see that

$$[A][\{y_{i,\mu} | i \in I, 1 \leqslant \mu \leqslant m\}] = [B];$$

indeed, for any $i, j \in I$, $1 \leqslant k \leqslant n$ and $1 \leqslant \mu \leqslant m$ we find $\varphi_j(a_k y_{i,\mu}) = \varphi_j(b_\mu)\varphi_j(a_{\nu(i)}^{-1} z_i a_k)$, and hence $\varphi_j(b_\mu) | \varphi_j(a_k y_{i,\mu})$ with equality if $i = j$ and $k = \nu(i)$.

CLAIM 3: For $A, B \in \mathscr{E}(H)$, $[A \cup B]$ is a GCD of $[A]$ and $[B]$ in D.

Indeed, if A, $B \in \mathscr{E}(H)$, then

$$\rho([A \cup B]) = \rho_0(A \cup B) = (\rho_0(A), \rho_0(B))_V = (\rho([A]), \rho([B]))_V,$$

and consequently $[A \cup B]$ is a GCD of $[A]$ and $[B]$, since ρ is a divisor homomorphism.

From CLAIM 3 we see that D is a GCD-monoid, and if $A = \{a_1, \ldots, a_n\} \in \mathscr{E}(H)$, then

$$[A] = ([a_1], \ldots, [a_n])_D = (\partial a_1, \ldots, \partial a_n)_D,$$

and therefore ∂ is a QDT. ∎

Theorem 4. *Let H be a monoid, $(\varphi_i : H \to V_i)_{i \in I}$ a defining family of essential valuations of finite character of H and $\psi : H \to V$ an essential valuation. Then there exists some $i \in I$ such that $\varphi_i^{-1}(1) \subset \psi^{-1}(1)$.*

Proof. By Theorem 3 H possesses a QDT, and therefore we may assume that $H \subset D$, where D is a GCD-monoid and $(H \hookrightarrow D)$ is a QDT. Let $\bar{\varphi}_i : D \to V_i$ and $\bar{\psi} : D \to V$ be the unique essential valuations satisfying $\bar{\varphi}_i | H = \varphi_i$ and $\bar{\psi} | H = \psi$ (according to Proposition 4). By Proposition 5, $(\bar{\varphi}_i)_{i \in I}$ is a defining family of D, and it suffices to show that $\bar{\varphi}_i^{-1}(1) \subset \bar{\psi}^{-1}(1)$ for some $i \in I$.

Assume to the contrary that, for all $i \in I$, there exists an element $a_i \in D$ such that $\bar{\varphi}_i(a_i) = 1$ and $\bar{\psi}(a_i) \neq 1$. We assert that there exists a finite subset $\{a_1, \ldots, a_n\} \subset \{a_i | i \in I\}$ such that $(a_1, \ldots, a_n)_D = 1$. Indeed, fix some $l \in I$ and consider the finite set $E = \{i \in I | \bar{\varphi}_i(a_l) \neq 1\} \cup \{l\}$; if $d \in D$ divides a_k for all $k \in E$, then $\bar{\varphi}_i(d) | \bar{\varphi}_i(a_k)$ for all $k \in E$ and $i \in I$, whence $\bar{\varphi}_i(d) = 1$ for all $i \in I$. Since $(\bar{\varphi}_i)_{i \in I}$ is a defining family, we conclude $d = 1$.

Having a subset $\{a_1, \ldots, a_n\}$ with the required properties we infer $\bar{\psi}(a_k) \neq 1$ for all $1 \leqslant k \leqslant n$ and therefore also $(\bar{\psi}(a_1), \ldots, \bar{\psi}(a_n))_V \neq 1$. This is a contradiction since $\bar{\psi}$ is a GCD-homomorphism by Proposition 1. ■

Recall that a family of monoid homomorphisms $(\varphi_i : H \to V_i)_{i \in I}$ is called *thin* if $i, j \in I$, $i \neq j$ implies $H_{\varphi_i} \not\subset H_{\varphi_j}$. A defining family of monoid homomorphisms $(\varphi_i)_{i \in I}$ is called *minimal* if for all $j \in I$ the family $(\varphi_i)_{i \in I \setminus \{j\}}$ is not a defining family.

Corollary 1. *Let H be a monoid and $\boldsymbol{\varphi}$ a defining family of essential valuations of finite character of H. If $\boldsymbol{\varphi}$ is thin, then it is a minimal defining family.*

Proof. Set $\boldsymbol{\varphi} = (\varphi_i)_{i \in I}$, fix some $j \in I$ and apply Theorem 4 to the family $((\varphi_i)_{i \in I \setminus \{j\}}$ and the homomorphism φ_j. Observe that $H_{\varphi_i} \subset H_{\psi}$ if and only if $\varphi_i^{-1}(1) \subset \psi^{-1}(1)$; cf. [3; Prop. 3.9]. ■

5. In this final section we present a result implying the Realization Theorem of Krull–Kaplansky–Jaffard–Ohm (cf. [9; §8] and [11; Theorem 2.1]).

Let R be an integral domain. We denote by $R^{\bullet} = R \setminus \{0\}$ the multiplicative monoid of R; then $R^{\times} = R^{\bullet \times}$. For a monoid homomorphism $\varphi : R^{\bullet} \to D$ we set $R_{\varphi} = R^{\bullet}_{\varphi} \cup \{0\}$; further we recall from [3; Definition 8.1] that φ is said to be *semiadditive*, if $\varphi(z) \mid \varphi(u)$ and $\varphi(z) \mid \varphi(v)$ implies $\varphi(z) \mid \varphi(u+v)$ for all $u, v \in R^{\bullet}$ such that $u + v \neq 0$.

Lemma 1. *If D is a* GCD*-monoid, then $\mathscr{Q}(D)$ is torsion free.*

Proof. Let $z \in \mathscr{Q}(D)$ and $n \in \mathbb{N}$ be such that $z^n = 1$. Then $z = a^{-1}b$, where $a, b \in D$, $(a, b)_D = 1$, and consequently $a^n = b^n$. By [5; Theorem 6.4], we infer $(a^n, b^n)_D = 1$ which implies $a^n = 1$ and hence $a = 1$ since D is reduced. ■

Lemma 2. *Let K be an integral domain and H a monoid such that the monoid ring $K[H]$ is an integral domain. Let $\theta : H \to V$ be a valuation and let $\psi : K[H]^{\bullet} \to V$ be defined by*

$$\psi(\zeta_1 c_1 + \cdots \zeta_n c_n) = (\theta c_1, \ldots, \theta c_n)_V,$$

if $c_1, \ldots, c_n \in H$ are distinct and $\zeta_1, \ldots, \zeta_n \in K^{\bullet}$. Then ψ is a monoid homomorphism.

Proof. An element $u \in K[H]^{\bullet}$ is called θ-homogeneous of degree $\gamma \in V$, if $u = \zeta_1 c_1 + \cdots + \zeta_n c_n$, where $\zeta_1, \ldots, \zeta_n \in K^{\bullet}$, $c_1, \ldots, c_n \in H$ are distinct and $\theta c_1 = \cdots = \theta c_n = \gamma$. Every $u \in K[H]^\bullet$ has a unique decomposition $u = u_1 + \cdots + u_r$, where u_i is θ-homomeneous of some degree $\gamma_i \in V$ such that $\gamma_1, \ldots, \gamma_r$ are distinct and $\gamma_1 | \gamma_2 | \cdots | \gamma_r$; let us call this decomposition the canonical one; clearly, $\psi(u) = \gamma_1$. If $u, v \in K[H]^\bullet$ have canonical decompositions $u = u_1 + \cdots + u_r$, $v = v_1 + \cdots + v_s$, then uv has canonical decomposition $uv = w_1 + \cdots + w_t$, where $w_1 = u_1 v_1$. This implies $\psi(u)\psi(v) = \psi(u_1)\psi(v_1) = \psi(w_1) = \psi(uv)$. ∎

Theorem 5. *Let H be a reduced monoid, ∂: $H \to D$ a* QDT, *K an integral domain, $R = K[H]$ the monoid ring and φ: $R^\bullet \to D$ defined by*

$$\varphi(\zeta_1 c_1 + \cdots + \zeta_n c_n) = (\partial c_1, \ldots, \partial c_n)_D,$$

if $c_1, \ldots, c_n \in H$ are distinct and $\zeta_1, \ldots, \zeta_n \in K^\bullet$. Then φ is a semiadditive monoid homomorphism, $A = R_\varphi$ is an integral domain and $A^\bullet/A^\times \simeq D$.

If $H = D$ and $\partial = \mathrm{id}_D$, then A is a Bezout domain.

Proof. Being injective, ∂ extends to an injective group homomorphism $\mathcal{Q}(\partial)$: $\mathcal{Q}(H) \to \mathcal{Q}(D)$. Since $\mathcal{Q}(D)$ is torsion free by Lemma 1, the same is true for $\mathcal{Q}(H)$, and hence $R = K[H]$ is an integral domain by [5; Theorem 8.1].

Let $(\rho_i: D \to V_i)_{i \in I}$ be a realization of D, $\varphi_i = \rho_i \circ \varphi$: $R^\bullet \to V_i$ and $\partial_i = \rho_i \circ \partial$: $H \to V_i$; every ∂_i is an essential valuation by Proposition 2 and hence a GCD-homomorphism by Proposition 1. If $u = \zeta_1 c_1 + \cdots + \zeta_n c_n \in R^\bullet$ (where $c_1, \ldots, c_n \in H$ are distinct and $\zeta_1, \ldots, \zeta_n \in K^\bullet$) then

$$\varphi_i(\zeta_1 c_1 + \cdots + \zeta_n c_n) = \rho_i((\partial c_1, \ldots, \partial c_n)_D) = (\partial_i c_1, \ldots, \partial_i c_n)_{V_i},$$

and hence φ_i is a monoid homomorphism by Lemma 2. Consequently,

$$\left(\prod_{i \in I} \varphi_i\right) = \left(\prod_{i \in I} \rho_i\right) \circ \varphi: R^\bullet \to \prod_{i \in I} V_i$$

is a monoid homomorphism, and since $\prod_{i \in I} \rho_i$ is injective, φ is also a monoid homomorphism.

It can be seen from the definition that φ is semiadditive, and therefore $A = R_\varphi$ is an integral domain by [3; Proposition 8.2]. Since ∂ is a QDT, φ is surjective and induces an isomorphism

$$\bar{\varphi}: A^\bullet/A^\times \to D$$

by [3; Lemma 2.2]. In particular, $A^\times = \mathrm{Ker}(\bar{\varphi}) = \mathrm{Ker}(\mathcal{Q}(\varphi))$ where $\mathcal{Q}(\varphi)$: $\mathcal{Q}(R^\bullet) \to \mathcal{Q}(D)$ is the unique group homomorphism extending φ.

It remains to prove that in the case $H = D$, $\partial = \mathrm{id}_D$, A is a Bezout domain. Let $J = {}_A\langle u_1, \dots, u_n \rangle$ be a finitely generated ideal of A and $\mathscr{Q}(\varphi)(u_\nu) = c_\nu \in D \subset A^\bullet$. Since $\mathscr{Q}(\varphi)|D = \mathrm{id}_D$, we obtain $\mathscr{Q}(\varphi)(c_\nu^{-1} u_\nu) = 1$ which implies $c_\nu^{-1} u_\nu \in A^\times$, and consequently $J = {}_A\langle c_1, \dots, c_n \rangle$. We set $c_0 = (c_1, \dots, c_n)_D$, and we assert that $J = c_0 A$. Indeed, if $c_\nu = c_0 c'_\nu$ with $c'_\nu \in D$, then $(c'_1, \dots, c'_n)_D = 1$ and therefore $c'_0 = c'_1 + \cdots + c'_n \in A^\times$. This implies

$$c_0 = c_0'^{-1}(c'_1 + \cdots + c'_n)c_0 = c_0'^{-1}(c_1 + \cdots + c_n) \in J$$

and hence $c_0 A \subset J$; the other inclusion is obvious. ■

In particular we have proved

Corollary 2 (Realization Theorem of Krull–Kaplansky–Jaffard–Ohm). *For every* GCD*-monoid D there exists a Bezout domain A such that $D \simeq A^\bullet/A^\times$.*

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